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# Optimal Takeover Contests with Toeholds\*

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## Abstract

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This paper characterizes how a target firm should be sold when the possible buyers (bidders) have prior stakes in its ownership (toeholds). We find that the optimal mechanism needs to be implemented by a non-standard auction which imposes a bias against bidders with high toeholds. This discriminatory procedure is such that the target's average sale price is increasing in both the size of the common toehold and the degree of asymmetry in these stakes. It is also shown that a simple mechanism of sequential negotiation replicates the main properties of the optimal procedure and yields a higher average selling price than the standard auctions commonly used in takeover battles.

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# 1 Introduction

Auctions in which bidders have stakes in the seller's surplus are not rare in the real world, as there are many examples that resemble a bidding competition with vertical toeholds. Takeover contests provide one of the clearest illustrations, since block shareholders compete among themselves or with an outside investor to gain the control of a company, while the minority shareholders play the role of *pure* sellers.<sup>1</sup>

In order to illustrate some of the features of takeover battles, consider the next real life example. In 2006, the Spanish tollway operator Europistas was the target of a takeover battle between two bidders. Firstly, the group Isolux Corsán submitted an offer for 100% of the ownership, consisting of 4.8 euros per share. At this stage, Cintra, one of the principal block shareholders of the target firm, attained an agreement with Isolux. According to the deal, Cintra committed itself to participate in this tender offer and sell irrevocably its 27.1 per cent stake for a price of 5.13 euros per share. In less than 24 hours, a second buyer emerged: a bidding consortium formed by the constructor conglomerate Sacyr Vallehermoso and three Basque saving banks grouped in the society Telekutxa. While Isolux Corsán was an outside bidder, Telekutxa held a 32.4 per cent stake in the capital of Europistas. The final tender offer of this consortium rose to 9.15 euros per share, that is, an improvement of 78.36% with respect to the first offer. This implied that Cintra was trapped in the pre-sale agreement reached with Isolux, which impeded it from taking advantage of the substantially better tender offer made by the consortium led by Sacyr. Finally, Cintra paid 131 million euros to Isolux as a compensation to recover its freedom to sell its stake to the bidding consortium, which was the winner of the contest and thus, took over Europistas.

This case highlights some interesting issues. First, unlike standard auctions, the presence of vertical toeholds introduces countervailing incentives on bidders because they can get a payoff not only when they win, but also when they lose the auction. In fact, since the losing bidder owns a proportion of the seller's surplus, he cares about the payment *received* by the seller. In the context of a takeover battle, as the winner bidder must buy all the shares, losing transforms a bidder with a toehold into a *minority seller*. This implies that, conditional on losing, a toehold induces a *more* aggressive bidding behavior. In addition, holding stakes in the target firm also means, by comparison with the outside bidders, lower costs

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<sup>1</sup>Other examples are creditors' bidding in bankruptcy auctions, or the negotiation of a partnership's dissolution. Also, a situation in which firms are related vertically, e.g. if a buyer firm hold shares in a supplier firm.

of overbidding when winning, as the amount of shares to be bought is smaller. Consequently, toeholds strengthen the standard incentive to increase bids present in any auction, but now with the intention of selling at a higher price.<sup>2</sup> Second, the aforementioned Europistas case illustrates the large costs that an incorrect choice of selling procedure may impose on the nonbidding shareholders' wealth.<sup>3</sup> Nonbidding shareholders of a target company - by means of the board of directors or a special committee - should therefore pay attention to the selling mechanism to be used.

The auction literature has studied takeovers using different valuation environments, but assuming always that signals are independently distributed. The main conclusion is that the more aggressive bidding behavior induced by toeholds leads to the break-down of the Revenue Equivalence Theorem (Myerson 1981, Riley and Samuelson 1981) even when bidders possess symmetric stakes. As a result, the equivalence between standard auctions no longer holds, as several papers have shown. In particular, Singh (1998), when analyzing a game in which a toehold bidder and an outside bidder compete to gain control of a company in a private values framework, has shown the superiority of the second-price auction over the first-price auction. The major insight stemming from his model is what he calls the *owner's curse*. According to this phenomenon, the higher aggressiveness of the toehold holder is so that in the second-price auction he is (rationally) willing to bid more than his valuation. Since this overbidding behavior is absent in the first-price auction due to the traditional trade-off present in this mechanism, the non-equivalence between both standard auctions emerges.<sup>4</sup> Bulow, Huang and Klemperer (1999) also study a two-bidder takeover contest, but under a common value set-up.<sup>5</sup> They compare the sealed-bid first-price and the ascending-price (equivalent to the second-price one) auctions in both the symmetric and the asymmetric cases. They show that with symmetric toeholds, the ascending auction performs better than the first-price auction in terms of the expected selling price per share. In contrast, when analyzing the asymmetric case, they find the opposite

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<sup>2</sup>In the context of the Europistas case, it is possible to conjecture about the source of the large price difference observed between the two offers. It seems plausible to argue that this gap reflected not only a higher valuation from the *toehold bidder* (the consortium headed by Sacyr), but also a more aggressive bidding behavior than that exhibited by the *outside bidder* (Isolux).

<sup>3</sup>The price difference of both tender offers (147 millions of euros) represented about eight times the annual net profits of Cintra.

<sup>4</sup>Ettinger (2002) confirms the dominance of the second-price auction over the first-price auction in terms of the expected sale price when buyers have *symmetric* stakes in the seller's surplus.

<sup>5</sup>They study takeovers among financial bidders for which, as the authors point out, the common values environment seems more appropriate.

result whenever toeholds are very asymmetric and sufficiently small.<sup>6</sup>

The current paper does also deal with the issue of how to run a takeover battle. But in sharp contrast with the previous literature, our work is, to the best of our knowledge, pioneering in that it adopts a *normative* approach rather than a positive one. That is, instead of taking a particular auction format as given for exogenous reasons, we analyze how the maximizing target price mechanism should be and how it could be implemented. To this end, our methodology follows the mechanism design approach, introduced by Myerson (1981), within an independent private values framework.

Two main features of our model are the possibility of asymmetry among bidders' toeholds and the existence of a bidder without toeholds (outside bidder). The analysis performed here is in close connection with Loyola (2007), a companion paper that characterizes the optimal mechanism in the presence of horizontal crossholdings, i.e., toeholds in other bidders' profits. In contrast with this case, we find that in the presence of vertical toeholds, the optimal allocation rule imposes no bias against any bidder as the presence of vertical toeholds only links the bidders' *payments*, but *not* the bidders' *valuations*. As a consequence, a maximizing revenue seller prefers a symmetric equilibrium even though buyers hold asymmetric stakes. It is shown however that this optimal rule needs to be implemented by a non-standard auction. In particular, we prove that the implementation is possible through a second-price auction augmented with a reserve price and a scheme of asymmetric payments. The latter includes a penalty against the winner (with respect to the non-toehold case) and a payment by the loser whenever he is a toehold bidder. The reason for this apparent contradiction between the allocation rule and the scheme of payments is the same as that behind the failure of the Revenue Equivalence Principle. That is, the presence of toeholds implies the impossibility of fully characterizing the revenues based only on the allocation rule and the payment made by the lowest-type bidder. With toeholds, the *entire* system of transfers plays a role to characterize revenues.

Our discriminatory policy has the following rationale. By imposing a heavier bias against the toehold bidder, the optimal mechanism extracts more surplus from the strongest player in the game. In the context of takeovers, this advantaged player corresponds to the raider who bids more aggressively due to his larger stake in the target. As a result, the discriminatory rule pays the seller, as we show that

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<sup>6</sup>These contrasting findings rest on two facts. First, the negative effect of the winner's curse on bidders' aggressiveness is larger in asymmetric ownership structures. Second, the first-price auction involves an allocation rule that is less sensitive to the distortions caused by the presence of toeholds.

the expected selling price is increasing in both the common toehold (the symmetric case) and in the degree of asymmetry in these stakes (the asymmetric case).<sup>7</sup> In addition, we show that a sequential negotiation procedure replicates the main properties of the optimal mechanism. This negotiation-based procedure sets an agenda of take-it-or-leave-it offers that gives priority to the more aggressive bidder, i.e., the toehold holder, and thus yields a higher target's expected sale price than both the first-price and the second-price auctions.<sup>8</sup> The last result is in line with the established superiority of sequential mechanisms which give priority to stronger bidders. Povel and Singh (2006), for instance, analyze takeover contests under a general value setting that allows both private and common value environments. They characterize the optimal selling procedure that a target company should design when it faces two outside bidders (without toeholds) who are asymmetrically informed. Interestingly, Povel and Singh also conclude about the optimality of imposing a heavier bias against the strongest bidder (the better-informed one) by means of a two-stage procedure. Similarly, Dasgupta and Tsui (2003) examine in an interdependent value setting the properties of the "matching auction", a sequential procedure where the first mover is also the strong bidder. In their model, the strong player can be either the larger-toehold bidder or the better-informed one. As with our sequential procedure, Dasgupta and Tsui also find that the matching auction allows the target's seller to obtain a higher expected transaction price than with the standard auctions, but only when asymmetry is sufficiently large. An important difference between the last two papers and ours, apart from the valuation environment adopted, lies in the mechanism itself, which implies bidders' participation strategies of different nature. Povel and Singh (2006) propose a hybrid sequential procedure that combines standard auctions and exclusive deals. Similarly, in the auction-based mechanism studied by Dasgupta and Tsui (2003), the first mover bidder *actively* follows a bid strategy, whereas the second one only decides whether or not match this bid. In contrast, our procedure is based upon a scheme of take-it or leave-it offers made by the seller so that all bidders are in some sense *passive* players.

This paper proceeds as follows. Section 2 sets up a model of takeover contests in the presence of toeholds. In Section 3, the optimal selling mechanism is

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<sup>7</sup>This revenue-increasing property of an optimal discriminatory policy has also been found in contests with asymmetric informed buyers (see Povel and Singh 2004, Povel and Singh 2006).

<sup>8</sup>In light of this finding, the Europistas case provides then a clear example of how things should *not* be done when selling a target firm in which one of the shareholders could become a bidder. Of course, in this case the nonbidding shareholder (Cintra) chose incorrectly to negotiate and close a deal first with the outside bidder (Isolux) instead of doing it previously with the toehold bidder (the consortium). In this paper we show that an appropriate sequential negotiation mechanism should take the opposed order of negotiations.

characterized and its main properties are established. In Section 4, we propose a simple negotiation procedure that replicates most of these properties. The next section compares this negotiation-based mechanism with the auction formats commonly used in practice. Finally, in Section 6 we conclude and stress some policy implications. All the proofs are collected in the Appendix.

## 2 The Model

The nonbidding shareholders of a target company (the *seller*), represented by the board of directors or a special committee, face a takeover threat from two possible risk-neutral buyers (the *bidders*). The value of the target to bidder  $i$  is  $t_i$ , which is private information, but it is common knowledge that it is independently and identically drawn according to c.d.f.  $F$  with support  $[\underline{t}, \bar{t}]$ , density  $f$  and hazard rate  $H(t_i) = f(t_i)/(1 - F(t_i))$ .<sup>9</sup> Denote the value that the initial shareholders assign to the target company by  $t_0$ , which is common knowledge and is here normalized to zero.<sup>10</sup>

A toehold of bidder  $i$  is defined as a partial participation of this bidder in the seller's surplus, or, equivalently, a partial participation in the ownership of the target company. We assume that bidder 1 has a larger initial stake in the seller's surplus than bidder 2. The parameter  $\phi_i$  represents the share of bidder  $i$  in the seller's surplus. Thus,  $(1 - \phi_1 - \phi_2)$  represents the participation of the seller in her surplus. Toeholds are assumed common knowledge, with  $1/2 > \phi_1 \geq \phi_2 \geq 0$ .<sup>11</sup>

We will also refer to the players as follows: a bidder with toehold as a *bidding shareholder* (or *toehold bidder*), a bidder without toehold as an *outside bidder* (or *non-toehold bidder*) and the seller as the *nonbidding shareholder*. Given this ownership structure, we interpret  $t_0$  as the common value that all shareholders assign to the firm when they own it partially. In other words,  $t_0$  represents the value that all shareholders assign to the firm under the current management, i.e., either before the takeover takes place or when this process is finally unsuccessful.

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<sup>9</sup>As it is standard in auction theory, we concentrate on the regular case, that is, increasing hazard rates.

<sup>10</sup>As we will see below, the seller may not be an exclusive initial owner.

<sup>11</sup>Notice that this formulation allows the presence of an outside bidder (non-toeholder), which is precisely the case analyzed in Section 5, given its predominance in actual takeovers. Bradley et al. (1988) find that 66% of the bidders in their sample of 236 successful tender offers have zero toeholds, while Betton and Eckbo (2000) find that 47% of initial bidders in their sample of over 1,300 tender offers (including failed ones) have zero toeholds (see Goldman and Qian (2005)).

In contrast, we understand  $t_i$  to be the private value that bidder  $i$  assigns to the target when he owns it fully. In consequence  $t_i$  can be interpreted as a private synergy that bidder  $i$  can exploit when he wins the contest and obtains absolute control of the company. It is also called the value "to run the firm".<sup>12</sup> Implicit in this interpretation is the assumption that the takeovers modeled in the present paper are not partial. That is, all shareholders must sell their stakes to the winning contestant (and he must buy it) according to the price stated by the contest's rules.

### 3 The Optimal Mechanism

Due to the revelation principle, we only need to focus on direct revelation mechanisms. Denote the vector of signals realizations by  $t$ , i.e.,  $t = (t_1, t_2)$ , and similarly, denote by  $t_{-i}$  the vector of signal realizations of all bidders except bidder  $i$ . Let  $T$  and  $T_{-i}$  be the support of  $t$  and  $t_{-i}$ , respectively.<sup>13</sup> Define  $p_i(t)$  as the probability with which the optimal mechanism allocates the target company to bidder  $i$ , conditional on the vector of reported signal realizations  $t$ , and, define  $x_i(t)$  as the transfer from bidder  $i$  to the seller, conditional on the same vector. Let  $Q_i(t_i)$  be bidder  $i$ 's conditional probability of winning given that his type is  $t_i$ , i.e.,  $Q_i(t_i) \equiv \int_{T_{-i}} p_i(t_i, t_{-i}) f(t_{-i}) dt_{-i}$ . Bidder  $i$ 's expected payoff, conditional on signal  $t_i$  and announcement  $\hat{t}_i$ , is then given by<sup>14</sup>

$$U_i(\hat{t}_i/t_i) \equiv \int_{T_{-i}} [(t_i p_i - (1 - \phi_i) x_i) + \phi_i x_j] f(t_{-i}) dt_{-i}$$

for all  $t_i, \hat{t}_i \in [\underline{t}, \bar{t}]$  and for  $i, j = 1, 2, i \neq j$ . We define bidder  $i$ 's truthtelling payoff as  $V_i(t_i) \equiv U_i(t_i/t_i)$  and the seller's expected revenue when all bidders report their true type as follows<sup>15</sup>

$$U_0 \equiv \sum_{i=1}^2 \int_T (1 - \phi_1 - \phi_2) x_i(t) f(t) dt. \quad (1)$$

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<sup>12</sup>Alternatively, since we have normalized  $t_0 = 0$ ,  $t_i$  can be interpreted as an incremental cash flow generated by the new control and management under bidder  $i$  (See Singh, 1998).

<sup>13</sup>In our set-up  $t_{-i}$  is just  $t_j$ . We have opted for the notation  $t_{-i}$  since the characterization of the optimal mechanism can be easily extended to the case of more than two bidders. For the three bidder case (two asymmetric toeholders and one outside bidder) the characterization can be obtained from the author upon request.

<sup>14</sup>For the sake of presentation, we have omitted the arguments of  $p_i$  and  $x_i$ , but it should be clear that  $p_i = p_i(\hat{t}_i, t_{-i})$  and  $x_i = x_i(\hat{t}_i, t_{-i})$ , for all  $i$ .

<sup>15</sup>This function is similar to that defined for the nonbidding shareholders by Bulow, Huang and Klemperer (1999) in the context of a takeover contest with *common* values.

Let us define  $c_i(t_i)$ , bidder  $i$ 's marginal revenue,<sup>16</sup> as

$$c_i(t_i) \equiv t_i - \frac{1}{H(t_i)} \text{ for all } i.$$

Following Myerson (1981) (see more details in the Appendix), it can be shown that the optimal mechanism solves the following problem:<sup>17</sup>

$$\max_{p_i, V_i(\underline{t})} \sum_{i=1}^2 \left[ -V_i(\underline{t}) + \int_T c_i(t_i) p_i(t) f(t) dt \right] \quad (2)$$

*s.t.*

$$V_i(\underline{t}) \geq 0, \text{ for all } i. \quad (3)$$

$$Q'_i(t_i) \geq 0 \text{ for all } t_i \in [\underline{t}, \bar{t}] \text{ and for all } i. \quad (4)$$

$$\sum_{i=1}^2 p_i(t) \leq 1 \text{ and } p_i(t) \geq 0, \text{ for all } i \text{ and for all } t \in T, \quad (5)$$

where (3) is a sufficient condition for bidder  $i$ 's participation constraint to hold, (4) is a sufficient condition for the incentive compatibility constraints of the bidders to hold and (5) corresponds to the feasibility constraints.

### 3.1 Optimal allocation rule

**Lemma 3.1** *The optimal mechanism sets  $V_i(\underline{t}) = 0$  and*

$$p_i(t) = \begin{cases} 1 & \text{if } c_i(t_i) > \max \{0, \max_{j \neq i} c_j(t_j)\} \\ 0 & \text{otherwise} \end{cases}$$

*for all  $i$ , and for all  $t \in T$ .*

Note that bidder  $i$ 's marginal revenue is larger than bidder  $j$ 's if and only if  $t_i > z_{ij}(t_j) \equiv c_i^{-1}(c_j(t_j))$  for all  $i \neq j$ . In addition, let us define  $t_i^* \equiv c_i^{-1}(0)$  as the threshold signal for which bidder  $i$ 's marginal revenue is larger than the seller's. Since  $c_i$  is well-behaved, so it is its inverse function, and thus it is equivalent to say that the optimal mechanism sets  $V_i(\underline{t}) = 0$  and

$$p_i(t) = \begin{cases} 1 & \text{if } t_i > \max \{t_i^*, \max_{j \neq i} z_{ij}(t_j)\} \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

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<sup>16</sup>Bulow and Roberts (1989) provide an interpretation of  $c_i(t_i)$  as the bidder  $i$ 's *marginal revenue*, instead of the bidder  $i$ 's *virtual valuation* concept defined by Myerson (1981).

<sup>17</sup>Notice that this problem is identical to the optimization program in Myerson (1981), who does *not* consider the presence of toeholds.



for all  $i$ , and for all  $t \in T$ .

Lemma 3.1 establishes that, in the presence of vertical toeholds, the optimal allocation rule is *not* a discriminatory one as the policy function satisfies that  $z_{ij}(t_j) = t_j$  as  $c_i(.) = c(.)$  for all bidders.<sup>18</sup> This implies that even though bidders possess asymmetric toeholds, it is revenue maximizing for the nonbidding shareholders to offer them the same chances of winning whenever they report the same signal value. This result is surprising because one would expect that, since a vertical toehold induces a more aggressive bidding behavior, the seller should take it into account to design the optimal rule. Our interpretation is that, as opposed to horizontal crossholdings (see Loyola 2006), vertical toeholds only impose links between the bidders' *payments*, but *not* between the bidders' *valuations*. Consequently, in the terminology of Bulow and Roberts (1989), the marginal revenue function (which depends only on valuations) is the same for all bidders. This implies that the seller perceives all bidders as symmetric players, and hence, it is optimal to impose no bias and to attain a symmetric equilibrium.

However, as we will see in the next subsection, this optimal symmetric equilibrium requires the seller to introduce an asymmetry into the payment scheme. The underlying rationale for this apparent contradiction between the allocation rule and the scheme of transfers is the same as the one behind the break-down of the Revenue Equivalence Principle. That is, when toeholds exist, revenues *do* depend on the entire payment scheme, not only on the transfers made by the lowest type bidder. As a result, it does not suffice to examine only the allocation rule to state the properties of the optimal mechanism. In fact, one needs to characterize the payment scheme fully as this is crucial in order to recognize the non-standard and discriminatory nature of the optimal selling procedure.

### 3.2 Implementation

Since all bidders provide the same marginal revenue, the implementation of the optimal allocation rule requires a scheme of payments that induce an efficient allocation, that is, one which guarantees that the target firm be awarded to the bidder who values it the most. Since we have assumed that players are asymmetric in their toeholds, and thus in their expected payoff functions, the only way to attain an efficient allocation is to design a scheme of “*personalized*” payments. This implies that we must rule out any standard auction, as it imposes symmetric payments on the players and thus results in an asymmetric and inefficient equilibrium.

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<sup>18</sup>In the terminology introduced by Bulow and Roberts (1989), all bidders exhibit the same marginal revenue function for the seller, who is interpreted as a monopolist.

This fact is formalized in the next corollary.

**Corollary 3.1** *A standard auction cannot implement the optimal selling mechanism.*

From the incentive compatible constraint, we show next that the optimal allocation rule can be implemented by a selling mechanism with an asymmetric scheme of transfers.

**Proposition 3.1** *In the presence of toeholds, the optimal mechanism can be implemented by a modified second price auction with a reserve price and a scheme of payments that includes a penalty against the winner and a payment by the loser. The scheme is the following one:*

$$x_i(t) = \begin{cases} z_i(t_{-i}) + [\delta_i - 1] z_i(t_{-i}) & \text{if } p_i(t) = 1 \\ \pi_i z_j(t_{-j}) & \text{if } p_i(t) = 0 \text{ and } p_j(t) = 1 \\ 0 & \text{otherwise} \end{cases}$$

for all  $i, j = 1, 2, i \neq j$ , and for all  $t \in T$ , where

$$\delta_i \equiv \frac{1 - \phi_j}{(1 - \phi_i - \phi_j)}, \quad \pi_i \equiv \frac{\phi_i}{(1 - \phi_i - \phi_j)},$$

and  $z_i(t_{-i}) = \inf \{s_i : c_i(s_i) \geq 0 \text{ and } c_i(s_i) \geq c_j(t_j)\}$ .

This scheme of payments has the following properties.

**Discriminatory policy with winning penalties and losing payments.** First, since  $z_i(t_{-i}) > 0$  and  $\delta_i \geq 1$ , this implies that when the winner is a bidder with toeholds, his payment has a *penalty* when compared to the payment he would make in case of holding no toeholds. This penalty is given by  $[\delta_i - 1] z_i(t_{-i})$ . Second, since  $z_j(t_{-j}) > 0$ , and  $\pi_i \geq 0$ , this means that when the loser is a bidder with an initial stake, his payment is *positive*. Third, from  $\phi_1 > \phi_2$ , it follows that  $\delta_1 > \delta_2$  and  $\pi_1 > \pi_2$ . Thus, it is clear that the scheme of transfers proposed imposes a discriminatory policy with a bias against the bidder with the largest initial stake.<sup>19</sup>

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<sup>19</sup>Moreover, this discriminatory policy gets exacerbated with the degree of asymmetry, as the gaps of both winning penalties and losing payments are increasing with the difference in toeholds.

**Truth-telling and efficient mechanism.** The discriminatory scheme of winning penalties and losing payments implies that the payoff of bidders 1 and 2 simplifies to

$$\pi_i(t_i) = \begin{cases} t_i - z_i(t_{-i}) & \text{if } p_i(t) = 1 \\ 0 & \text{otherwise} \end{cases}$$

The scheme of transfers therefore induces symmetric objective functions for all bidders, as in the standard problem when there are no toeholds (see Myerson 1981).

**Average sale price increasing with common toeholds and asymmetry.** First, let  $\Pi_0^*$  be the seller's expected revenue under the optimal mechanism, and hence, define  $\rho_0^* \equiv \Pi_0^*/(1 - \phi_1 - \phi_2)$ , the average sale price under the same procedure. From (1) and Proposition 3.1, it follows directly that  $\Pi_0^*$ , and thus  $\rho_0^*$ , are increasing with both the winning penalty and the losing payment. Second, consider the symmetric toeholds case (i.e.  $\phi_1 = \phi_2 = \phi > 0$ ). In this case, both the winner's penalty and the loser's payment are increasing in the common toehold, as it is easy to check that  $\partial\delta_i/\partial\phi > 0$  and  $\partial\pi_i/\partial\phi > 0$  for all  $i$ . All of this implies that, at the optimal mechanism, the seller's expected revenue (and thereby, the average sale price) is *increasing* with the size of common toeholds. Finally, consider the asymmetric toeholds case (i.e.  $\phi_1 > \phi_2 > 0$ ). Let us define  $\varepsilon \equiv \phi_1 - \phi_2$  so that the parameters of the winning penalty and the losing payment can be rewritten as

$$\begin{aligned} \delta_1 &= \frac{1 - \phi_2}{1 - 2\phi_2 - \varepsilon}, \delta_2 = \frac{1 - \phi_2 - \varepsilon}{1 - 2\phi_2 - \varepsilon} \\ \pi_1 &= \frac{\phi_2 + \varepsilon}{1 - 2\phi_2 - \varepsilon}, \pi_2 = \frac{\phi_2}{1 - 2\phi_2 - \varepsilon} \end{aligned}$$

Hence, it is easy to verify that  $\partial\delta_i/\partial\varepsilon > 0$  and  $\partial\pi_i/\partial\varepsilon > 0$  for all  $i$ . Therefore, the optimal mechanism is such that the seller's expected revenue (and thus, the average sale price) is *increasing* with the degree of asymmetry in toeholds. All of this implies that a discriminatory policy pays to the seller.

## 4 A sequential negotiation procedure

In this section we show that a simple sequential negotiation procedure replicates the main properties of the optimal mechanism. The negotiation procedure works as follows:

### Stage I

**I.1.** The seller makes a take-it-or-leave-it offer  $\rho_1$  to bidder 1, where the offer  $\rho_i$  is the price to be paid by bidder  $i$  for the target shares.

**I.2.** Bidder 1 accepts or rejects this offer. If he accepts, the target is sold to him and the game is over. If bidder 1 rejects the exclusive deal, negotiation moves to the next round.

## Stage II

**II.1.** The seller makes a new take-it-or-leave-it offer  $\rho_2$  to bidder 2.

**II.2.** Bidder 2 accepts or rejects this offer. If he accepts, the target is sold to him. Otherwise, the target company remains under the current ownership structure and management.

The next proposition illustrates the discrimination policy resulting from the negotiation procedure for the uniformly distributed valuations case.<sup>20</sup>

**Proposition 4.1** *Suppose that  $t_i$  is uniformly distributed in the interval  $[0, 1]$  for all  $i = 1, 2$ . At the Subgame Perfect Nash equilibrium of the game induced by the sequential negotiation procedure, it is optimal for the seller to set  $\rho_1^* > \rho_2^*$  for all  $\phi_1 \geq \phi_2 \geq 0$ .*

With sequential negotiations the sale price charged to the first bidder is higher than the one charged to the second bidder. As the first-mover is the buyer with the highest toehold, the sequential mechanism discriminates against him. Moreover, the degree of this bias *increases* with the degree of asymmetry in the toeholds. More precisely, if we define the degree of asymmetry by  $\varepsilon \equiv \phi_1 - \phi_2$ , then the difference in prices offered by the seller, i.e.,  $\Delta\rho(\phi_2, \varepsilon) \equiv \rho_1^* - \rho_2^*$ , is increasing in  $\varepsilon$ . To see this note that

$$\Delta\rho(\phi_2, \varepsilon) \equiv \rho_1^* - \rho_2^* = \frac{1 - 2\phi_2 + 4\varepsilon}{8(1 - (\phi_2 + \varepsilon))(1 - \phi_2)} \text{ so that } \partial\Delta\rho(\phi_2, \varepsilon)/\partial\varepsilon > 0.$$

Note also that  $\Delta\rho(\phi_2, \varepsilon)$  is strictly increasing in  $\phi_1$  and strictly decreasing in  $\phi_2$ , with  $\Delta\rho(\phi_2, \varepsilon)$  strictly increasing in  $\phi_2$  for fixed and given  $\varepsilon$ . Hence, the negotiation procedure highlights the importance of establishing an asymmetric scheme of payments, as the price charged to the high-toehold bidder exceeds that of the low-toehold one, and this bias is larger when the ownership stakes become more asymmetric.

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<sup>20</sup>For simplicity and without loss of generality, all the results in the paper are henceforth stated assuming uniformly distributed valuations on the unitary interval.

To analyze whether this price discrimination policy pays to the seller we must look at the average sale price delivered by the equilibrium of the sequential procedure. Let  $\Pi_0^{SN}$  be the seller's expected revenue under the sequential procedure, and consequently, define  $\rho_0^{SN} \equiv \Pi_0^{SN}/(1 - \phi_1 - \phi_2)$ , the average sale price under the same mechanism.<sup>21</sup> Rewriting  $\rho_0^{SN}$  in terms of  $\varepsilon = \phi_1 - \phi_2$ , it follows that

$$\rho_0^{SN} = \frac{1}{16(1 - \phi_2)^2} \left[ \frac{(5 - 6\phi_2)^2}{4(1 - \phi_2 - \varepsilon)} + \phi_2 + \varepsilon \right].$$

It is easy to verify that  $\partial \rho_0^{SN}/\partial \varepsilon > 0$  for all  $\phi_2, \varepsilon \in (0, 1/2)$  so that the average sale price is *increasing* in the *degree of asymmetry*. This result is displayed in Figure 1.

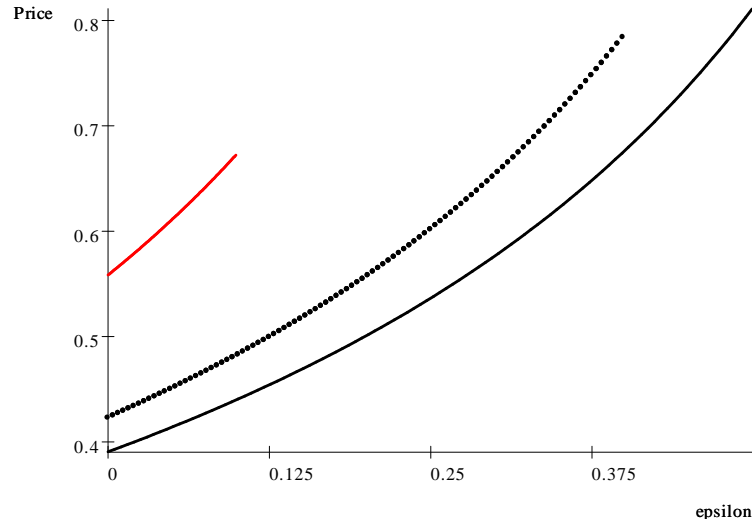


Figure 1. Average sale price from the sequential negotiation mechanism with two bidders and  $\phi_1 > \phi_2 \geq 0$ , for  $\phi_2 = 0$  (solid line),  $\phi_2 = .1$  (dot line) and  $\phi_2 = .4$  (dash line).

Furthermore, similarly to the optimal mechanism in the *symmetric* case, the afore-defined sequential procedure yields an average sale price which is also *increasing* in the *common toehold*. In fact, when  $\phi_1 = \phi_2 = \phi$ , it is possible to check that  $\partial \rho_0^{SN}/\partial \phi > 0$  for all  $\phi \in (0, 1/2)$ , as it is illustrated in Figure 2.

<sup>21</sup>See the Appendix (Proof of Proposition 4.1) for details on how this average price is computed.

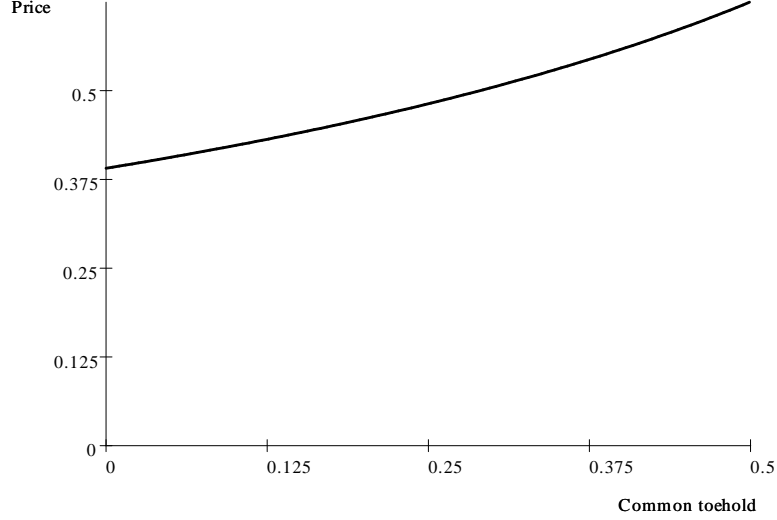


Figure 2. Average sale price from the sequential negotiation mechanism with two bidders for  $\phi_1 = \phi_2 = \phi \geq 0$ .

Notice however, that unlike the optimal mechanism described in the previous section, the sequential procedure always discriminates against the first-mover bidder, even if the toeholds are symmetric or zero. In fact, as the proof of Proposition 4.1 establishes, the prices charged to both players in the symmetric case (i.e.,  $1/2 > \phi_1 = \phi_2 = \phi \geq 0$ ) satisfy the following inequality

$$\rho_1^* = \frac{5 - 6\phi}{8(1 - \phi)^2} > \frac{1}{2(1 - \phi)} = \rho_2^*$$

In addition, the different priorities given by the negotiation timetable to different buyers implies that, unlike the optimal procedure, the sequential mechanism may be ex post *inefficient*.

In sum, and despite these differences, our sequential procedure replicates the two most important properties of the optimal mechanism: the expected selling price is increasing in both the common toehold and the degree of asymmetry in the initial stakes held by bidders.

## 5 Sequential procedure vs. auctions

Although there is not a specific practice to sell a company, sometimes the legal framework implicitly induces the board of directors to conduct an auction among

the raiders.<sup>22</sup> The underlying rationale behind this recommendation is the idea that an auction run with several bidders at once offers a more competitive environment than a negotiation held with a single buyer at each round. Nevertheless, and despite this idea, the coexistence of both types of mechanisms in real world takeover processes has been widely documented.<sup>23</sup> In this Section we compare the sequential procedure to the auction formats commonly used in practice from the nonbidding shareholders' point of view. We show that the nonbidding shareholders benefit from the discrimination policy to the extent that the sequential procedure generates a higher expected selling price than both the first-price and the second-price auctions.

We analyze here two ownership structures in which this result holds: (i) the symmetric case, i.e.  $\phi_1 = \phi_2 = \phi \geq 0$ , and (ii) a particular asymmetric case in which there are two classes of bidders: one toeholder and one outsider, i.e.,  $\phi_1 > \phi_2 = 0$ .<sup>24</sup> For both of these ownership structures, the literature provides a ranking between the first and second price formats. In the second-price auction, and for both ownership environments, the toehold bidder exhibits the *owner's curse*, an overbidding behavior according to which the equilibrium bid exceeds his valuation. This overbidding phenomenon is however not present in the case of the outside raider, as bidding his true valuation continues to be a dominant strategy for him. In contrast, given the traditional bidding trade-off present in the first-price auction, the owner's curse is absent in this selling format. Because of this, the second-price auction outperforms the first-price auction in terms of revenue, in both the symmetric and asymmetric structures.<sup>25</sup> As a result, it suffices to compare the selling price generated by the sequential mechanism with that generated by the second-price auction.

The following auxiliary result characterizes the expected selling price in the second-price auction.

**Lemma 5.1** *Let  $\rho_0^{SPA}$  be the average sale price resulting from the second-price auction. Then,*

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<sup>22</sup>For instance, the Delaware law in the US establishes that the board must act as "auctioneers charged with getting the best price for the stock-holders at a sale of the company". See also Cramton and Schwartz (1991).

<sup>23</sup>See the evidence provided by Boone and Mulherin (2003), Boone and Mulherin (2004), Povel and Singh (2006), and Bulow and Klemperer (2007).

<sup>24</sup>As the evidence presented by Bradley et al. (1988), Betton and Eckbo (2000), and Betton Eckbo and Thorburn (2005) suggests, the presence of an outside bidder is very common in actual takeovers.

<sup>25</sup>Ettinger (2002) performs this comparison for the symmetric case, and Ettinger (2005) does it for the specific asymmetric environment analyzed here.

(1) In the symmetric case, this price is given by

$$\rho_0^{SPA} = \frac{(1+2\phi)(1-\phi)}{(1-2\phi)(1+\phi)} - \frac{2}{3(1-2\phi)}$$

(2) In the asymmetric case, it corresponds to

$$\rho_0^{SPA} = \frac{1}{1-\phi_1} \left[ \frac{\phi_1}{\phi_1+1} - \frac{5}{6}\phi_1 - \frac{1}{2\phi_1+2} + \frac{2}{3\phi_1+3} + \frac{1}{6} \right]$$

Now, we establish the predominance of our sequential mechanism over the auction formats commonly used in practice, irrespective of the degree of symmetry in toeholds.

**Proposition 5.1** *The sequential procedure generates a higher average sale price than both the first-price and the second-price auctions, no matter the degree of asymmetry.*

As mentioned in the previous section, the sequential procedure always discriminates against the first-mover bidder. This fact implies that it yields a larger expected sale price than both auction formats in the symmetric case, even when there are no toeholds at all. The average sale price comparison for the symmetric case between the second-price auction and our sequential mechanism is depicted in Figure 3. Note from the figure that the second-price auction induces a *concave* average sale price whereas the negotiation procedure exhibits a *convex* one. As a result, the price gap between both mechanisms is larger when the toehold becomes sufficiently low or sufficiently high. The difference attains its minimum for values around .25.



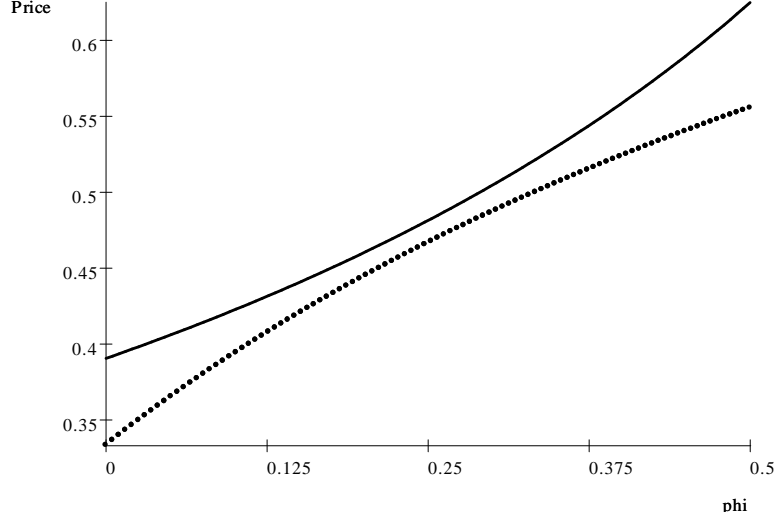


Figure 3. Average sale price from the sequential negotiation mechanism (solid line) and the SPA (dot line) with two bidders for  $\phi_1 = \phi_2 = \phi \geq 0$ .

Furthermore, the superiority of our sequential mechanism over auctions is exacerbated in the asymmetric case, as the discriminatory policy involves a sequence of negotiations with a pecking order consistent with the aggressiveness of each buyer (see Figure 4).

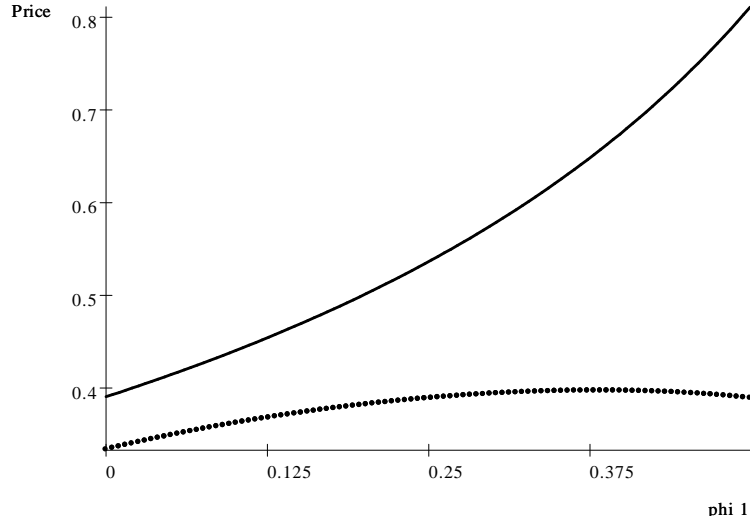


Figure 4. Average sale price from the sequential negotiation mechanism (solid line) and the SPA (dot line) with two bidders for  $\phi_1 > \phi_2 = 0$ .

Notice that the clear advantage of the sequential mechanism becomes larger when the degree of asymmetry (represented in this case by  $\phi_1$ ) increases. This is a con-

sequence of the fact that whereas  $\rho_0^{SN}$  is always an increasing and convex function in  $\phi_1$ ,  $\rho_0^{SPA}$  is a concave function and an increasing one *only* for a sufficiently low degree of asymmetry (for all  $\phi_1 < .38$ ).

This last result is formalized in the following statement.

**Corollary 5.1** *The larger the degree of asymmetry, the better the sequential procedure when compared with both the first-price and the second-price auctions.*

Finally, let us mention that our results here are in line with the well-established supremacy of sequential mechanisms which give higher priority to stronger bidders.<sup>26</sup> Accordingly, and in contrast with the standard auction formats, the particular order of negotiations involved in our procedure allows for an exploitation of the higher aggressiveness of raiders with larger stakes in the target.

## 6 Concluding Remarks

We have characterized how a target firm should be sold when bidders possess prior stakes in its ownership. This optimal mechanism corresponds to a non-standard auction with a scheme of asymmetric payments that imposes a bias against toeholders. The rationale of such a discriminatory policy is the fact that a standard mechanism is unable to induce a symmetric and efficient allocation rule, as it preserves the initial advantage of toehold bidders. In contrast, a scheme of asymmetric winning penalties and losing payments allows both to take advantage of the higher aggressiveness of toeholders and to go back to a symmetric environment.

The presence of losing payments in the optimal procedure is in line with similar results found in the literature devoted to characterizing optimal auctions when externalities exist. For instance, Goeree et al. (2005) show that the positive externalities present in fund-raising activities lead to discarding winner-pay auctions in favor of all-pay formats. In a result reminiscent of ours, they establish the optimality of an auction with a reserve price and payments by the losers - a mix between participation fees and an all-pay auction run in a subsequent stage-, which depend on the degree of the externality. Moreover, Goeree et al. (2005) emphasize that some characteristics of this optimal procedure are present in the

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<sup>26</sup>See Povel and Singh (2006) and Dasgupta and Tsui (2003).

procedures used for raising funds in the real world. As a consequence, the characteristics of our non-standard auction in the takeover case are not far from those exhibited by the optimal procedure in other contests with externalities.

We have also demonstrated that the nonbidding shareholders benefit from the discriminatory mechanism, as the target average sale price is increasing in both the common toehold and the degree of asymmetry in these stakes. The latter finding is in sharp contrast with the properties of standard auction formats in takeover battles, which then lead to opposite policy implications. For instance, Bulow, Huang and Klemperer (1999) show that in general the asymmetry in toeholds lowers prices in common-value ascending auctions. As a result, they recommend the "level the playing field" practice, according to which it may be revenue increasing to sell toeholds very cheaply to the buyer with the smaller stake in the target. On the contrary, our normative approach suggests that the seller should follow strategies with the aim of preserving this asymmetry. Accordingly, the board of directors should block or discourage the entrance of new shareholders suspected of becoming competitors against the incumbent toehold in a future takeover battle.

As an alternative to the optimal non-standard auction-based mechanism, we have proposed a simpler and realistic negotiation procedure that replicates the main properties of the first one. This mechanism contains a timetable that gives priority to the higher-toehold bidders, but charges higher prices to them. Such a negotiation-based procedure shares some features of other selling procedures already considered in the literature. In particular, it balances out properly the trade-off between creation and extraction of value caused by the implicit *threats* involved in the sequential nature of the negotiation process. This characteristic is also present in the posted-price rule discussed by Campbell and Levin (2006) in an environment with interdependent valuations. These authors find conditions under which a hybrid mechanism of a posted-price rule and a random rationing may outperform standard auctions. This fact occurs essentially when the increase of all buyers' willingness to pay offsets the losses stemming from ex post inefficient allocations. Similarly, in the context of our paper, the individual and sequential feature of the negotiation scheme imposes costs and benefits on nonbidding shareholders. On the one hand, the expected target price decreases due to both less competition and less efficiency. On the other hand, the higher priority given to the high-toehold bidder increases his willingness to pay, as the opportunity of winning the contest emerges even though his value may be lower than the small-toehold bidder's one. We have proved that the last effect dominates the shortcomings, therefore keeping open the ongoing debate on auctions versus negotiations in takeover wars.

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## 8 Appendix

### Appendix A: The optimal mechanism problem.

The optimal mechanism solves the following problem:

$$\begin{aligned} \max_{x_i \in \mathbb{R}, p_i \in [0,1]} U_0 \\ s.t. \end{aligned} \tag{7}$$

$$V_i(t_i) \geq 0 \quad \forall t_i \in [\underline{t}, \bar{t}], \quad i = 1, 2 \tag{8}$$

$$V_i(t_i) \geq U_i(\hat{t}_i/t_i) \quad \forall t_i, \hat{t}_i \in [\underline{t}, \bar{t}], \quad i = 1, 2 \tag{9}$$

$$\sum_{i=1}^2 p_i(t) \leq 1 \text{ and } p_i(t) \geq 0, \quad i = 1, 2, \forall t \in T \tag{10}$$

where (7) is the seller's expected revenue, (8) is bidder  $i$ 's participation constraint, (9) represents the incentive compatibility constraints of the bidders and (10) cor-

responds to the feasibility constraints.<sup>27</sup> From Myerson (1981), standard substitutions and computations lead to state the equivalence between the incentive compatibility constraints and the following two conditions

- (i)  $\frac{\partial V_i(t_i)}{\partial t_i} = Q_i(t_i)$
- (ii)  $\frac{\partial Q_i(t_i)}{\partial t_i} \geq 0$

These conditions allow to replace (9) by (ii) and

$$V_i(t_i) = V_i(\underline{t}) + \int_{\underline{t}}^{t_i} Q_i(s_i) ds_i. \quad (11)$$

Similarly, (8) is guaranteed to hold if  $V_i(\underline{t}) \geq 0$  for all  $i$ . Hence, straightforward computations allow us to rewrite the seller's expected payoff and to simplify the maximization problem as presented in Section 3.

## Appendix B: Proofs.

**Proof of Lemma 3.1.** From (2), it is in the seller's interest to make  $V_i(\underline{t}) = 0$  for all  $i$  because  $V_i(\underline{t}) > 0$  is suboptimal and setting  $V_i(\underline{t}) < 0$  violates the Participation Constraint. Moreover,  $H'(t_i) > 0$  implies that  $c'_i(t_i) > 0$  and thereby  $\partial p_i(t)/\partial t_i \geq 0$ , so that  $Q'_i(t_i) \geq 0$  is satisfied for all  $i$ . Finally, since  $t_0 = 0$ , the optimal allocation rule is found by comparing for a given  $t = (t_1, t_2)$  the terms  $c_i(t_i)$ , whenever they are positive. The solution sets then  $p_i(t) = 1$  iff  $c_i(t_i) > \max \{0, \max_{j \neq i} c_j(t_j)\}$ . ■

**Proof of Proposition 3.1.** For any vector  $t_{-i}$  consider

$$z_i(t_{-i}) = \inf \{s_i : c_i(s_i) \geq 0 \text{ and } c_i(s_i) \geq c_j(t_j) \text{ for all } j \neq i\}$$

for all  $i$ , i.e., the infimum of all winning values for  $i$  against  $t_{-i}$ . Then, in equilibrium

$$p_i(s_i, t_{-i}) = \begin{cases} 1 & \text{if } s_i > z_i(t_{-i}) \\ 0 & \text{if } s_i < z_i(t_{-i}) \end{cases} \quad (12)$$

and

$$\int_{\underline{t}}^{t_i} p_i(s_i, t_{-i}) ds_i = \begin{cases} t_i - z_i(t_{-i}) & \text{if } t_i \geq z_i(t_{-i}) \\ 0 & \text{if } t_i < z_i(t_{-i}) \end{cases} \quad (13)$$

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<sup>27</sup>Following Jehiel, Moldovanu and Stachetti (1996) and (1999), it is possible to show that the optimal threat for the non-participating bidder is that the target remains under the current management and control. As a result, the outside utility for the lowest-type bidder is the same for all buyers (toeholders and outsiders), and so, it can be normalized to zero (see Loyola 2007, Section 3).

for all  $i$ . Substitute  $Q_i(s_i)$  into (11), change the order of integration and substitute  $V_i(t_i)$ . After rearranging, we obtain that the truthtelling payoff of the bidder with the lowest signal can be written as

$$V_i(\underline{t}) = \int_{T_{-i}} \{t_i p_i(t) - [1 - \phi_i] x_i(t) + \phi_i \sum_{j \neq i} x_j(t) - \int_{\underline{t}}^{t_i} p_i(s_i, t_{-i}) ds_i\} f(t_{-i}) dt_{-i} \quad (14)$$

for all  $i$  and  $t_i \in [\underline{t}, \bar{t}]$ . Since it is optimal  $V_i(\underline{t}) = 0$  for all  $i$ , then sufficient conditions for (14) to hold are:

$$t_i p_i(t) - [1 - \phi_i] x_i(t) + \phi_i \sum_{j \neq i} x_j(t) = \int_{\underline{t}}^{t_i} p_i(s_i, t_{-i}) ds_i$$

for all  $i$  and for all state  $t = (t_i, t_{-i})$ . If we fix a particular state  $t = (t_i, t_{-i})$ , three cases are possible: (i) a winning bidder exists different from bidder 3, (ii) bidder 3 is the winner, and (iii) the object is not awarded to any bidder. Applying (12) and (13), the solution of this system of equations for the three cases yields the desired scheme of asymmetric payments. ■

**Proof of Proposition 4.1.** Using backward induction, we first characterize the Nash equilibrium resulting from Stage II. In this stage, bidder 2 accepts the offer if  $t_2 - (1 - \phi_2)\rho_2 > 0$ , i.e., if  $t_2 > (1 - \phi_2)\rho_2$ , and rejects otherwise. The seller's problem is hence

$$\max_{\rho_2} [(1 - \phi_1 - \phi_2)\rho_2] [1 - (1 - \phi_2)\rho_2],$$

whose solution is given by  $\rho_2^* = 1/2(1 - \phi_2)$ . The optimal seller's expected revenue from this stage is equal to  $(1 - \phi_1 - \phi_2)/4(1 - \phi_2)$ .

In stage I.2, bidder 1 accepts any seller's offer if his expected payoff is larger than the expected payoff at the equilibrium of stage II. That is, if  $t_1 - (1 - \phi_1)\rho_1 > E_{t_2}[\phi_1 \rho_2^*] = \phi_1/4(1 - \phi_2)$ , which is equivalent to the condition  $t_1 > (\phi_1/4(1 - \phi_2)) + (1 - \phi_1)\rho_1$ . Thus, the seller's optimal offer is characterized by

$$\begin{aligned} \rho_1^* = & \arg \max_{\rho_1} (1 - \phi_1 - \phi_2)\rho_1 \left[ 1 - \frac{\phi_1}{4(1 - \phi_2)} - (1 - \phi_1)\rho_1 \right] \\ & + \frac{1 - \phi_1 - \phi_2}{4(1 - \phi_2)} \left[ \frac{\phi_1}{4(1 - \phi_2)} + (1 - \phi_1)\rho_1 \right]. \end{aligned}$$

The solution is given by  $\rho_1^* = (5 - 6\phi_2)/8(1 - \phi_1)(1 - \phi_2)$ , which yields an optimal seller's expected revenue equal to

$$\Pi_0^{SN} = \frac{(1 - \phi_1 - \phi_2)}{16(1 - \phi_2)^2} \left[ \frac{(5 - 6\phi_2)^2}{4(1 - \phi_1)} + \phi_1 \right],$$

and an average sale price equal to

$$\rho_0^{SN} \equiv \Pi_0^{SN}/(1 - \phi_1 - \phi_2) = \frac{1}{16(1 - \phi_2)^2} \left[ \frac{(5 - 6\phi_2)^2}{4(1 - \phi_1)} + \phi_1 \right]. \quad (15)$$

Since  $1/2 > \phi_1 \geq \phi_2 \geq 0$ , it is simple to verify that

$$\rho_1^* = \frac{5 - 6\phi_2}{8(1 - \phi_1)(1 - \phi_2)} \geq \frac{5 - 6\phi_2}{8(1 - \phi_2)^2} > \frac{1}{2(1 - \phi_2)} = \rho_2^*$$

which proves the statement of the proposition. ■

**Proof of Lemma 5.1.** Since that the *asymmetric* case is the most general one, we first prove the second part of the proposition. In the second-price auction, bidder 2's payoff function, when his signals is  $t_2$  and he behaves as if it were  $\hat{t}_2$ , is given by

$$\pi_2(t_2, \hat{t}_2) = \max_{\hat{t}_2} \int_0^{b_1^{-1}(b_2(\hat{t}_2))} (t_2 - b_1(t)) dt, \quad (16)$$

that is, the traditional payoff function in a second-price auction without toeholds. Consequently, it follows that  $b_2(t_2) = t_2$ . Given the bid strategies  $b_1(\cdot)$  and  $b_2(t_2) = t_2$ , bidder 1's optimal choice of  $\hat{t}_1$  when he observes  $t_1$  is obtained by maximizing his expected profits

$$\pi_1(t_1, \hat{t}_1) = \max_{\hat{t}_1} \int_0^{b_1(\hat{t}_1)} (t_1 - (1 - \phi_1)t) dt + \phi_1 \int_{b_1(\hat{t}_1)}^1 b_1(\hat{t}_1) dt. \quad (17)$$

From Ettinger (2005), bidder 1's equilibrium bid is given by

$$b_1(t_1) = \frac{\phi_1}{1 + \phi_1} + \frac{t_1}{1 + \phi_1}.$$

Now, in order to compute the seller's revenues, let us define  $\psi_j(t_i)$ , the equilibrium correspondence function, such that  $b_i(t_i) = b_j(\psi_j(t_i))$  for all  $i, j = 1, 2$ . Applying the definition of  $\psi_j(\cdot)$  to the equilibrium bid strategies yields

$$\psi_2(t_1) = \frac{\phi_1}{1 + \phi_1} + \frac{t_1}{1 + \phi_1}, \quad (18)$$

$$\psi_1(t_2) = -\phi_1 + t_2(1 + \phi_1). \quad (19)$$

Appealing to the Envelope Theorem, and using the fact that  $\psi_2(\cdot) = b_1(\cdot)$  and  $\psi_1(\cdot) = b_1^{-1}(b_2(\cdot))$ , it can be verified that  $\frac{d\pi_i(t_1, \hat{t}_i)}{dt_i} = \psi_j(t_i)$ , which implies

$$\pi_i(t_i) = \pi_i(1) - \int_{t_i}^1 \psi_j(t) dt \quad (20)$$



for all  $i, j = 1, 2$ . Evaluating  $t_i = 1$  in (16) and (17), and using the fact that in equilibrium  $\psi_j(\widehat{t}_i) = \psi_j(t_i)$  and  $\psi_j(1) = 1$ , it can be shown that

$$\pi_1(1) = 1 - \frac{1 - \phi_1}{2} \quad (21)$$

$$\pi_2(1) = \frac{1}{2(1 + \phi_1)}. \quad (22)$$

Substituting (18), (19), and the results (21) and (22) into (20), bidder  $i$ 's interim payoff becomes

$$\begin{aligned} \pi_1(t_1) &= 1 - \frac{(1 - \phi_1)}{2} - \frac{(1 - t_i^2)}{2(1 + \phi_1)} - \frac{\phi_1(1 - t_1)}{(1 + \phi_1)} \\ \pi_2(t_2) &= \frac{1}{2(1 + \phi_1)} - 1 + \frac{1 + \phi_1}{2} + \frac{(1 + \phi_1)t_2^2}{2} - \phi_1 t_2. \end{aligned}$$

After taking expectations, bidder  $i$ 's ex-ante payoff is given by

$$\begin{aligned} \Pi_1 &= 1 - \frac{(1 - \phi_1)}{2} - \frac{1}{3(1 + \phi_1)} - \frac{\phi_1}{2(1 + \phi_1)} \\ \Pi_2 &= \frac{1}{2(1 + \phi_1)} + \frac{(1 + \phi_1)}{6} - \frac{1}{2} \end{aligned}$$

The nonbidding shareholders' expected revenues are then given by

$$\begin{aligned} \Pi_0^{SPA} &= \left[ \int_0^1 t_1 \int_0^{\psi_2(t_1)} dt_2 dt_1 + \int_0^1 t_2 \int_0^{\psi_1(t_2)} dt_1 dt_2 \right] - \Pi_1 - \Pi_2 \\ &= \left[ \int_0^1 t_1 \psi_2(t_1) dt_1 + \int_0^1 t_2 \psi_1(t_2) dt_2 \right] - \Pi_1 - \Pi_2 \\ &= \frac{\phi_1}{\phi_1 + 1} - \frac{5}{6}\phi_1 - \frac{1}{2\phi_1 + 2} + \frac{2}{3\phi_1 + 3} + \frac{1}{6} \end{aligned}$$

and the average selling price is

$$\rho_0^{SPA} \equiv \Pi_0^{SPA} / (1 - \phi_1) = \frac{1}{1 - \phi_1} \left[ \frac{\phi_1}{\phi_1 + 1} - \frac{5}{6}\phi_1 - \frac{1}{2\phi_1 + 2} + \frac{2}{3\phi_1 + 3} + \frac{1}{6} \right].$$

We now turn to demonstrate the statement for the *symmetric* case. From Proposition 1 in Ettinger (2002), the second-price auction equilibrium bid is given by

$$b_i(t_i) = \frac{\phi}{1 + \phi} + \frac{t_i}{1 + \phi}$$

for all  $i$ . Hence,  $\psi_2(t) = \psi_1(t) = t$  for all  $t$ . Applying the same line of reasoning used in the asymmetric case, it can be verified that the seller's expected revenues are given by

$$\begin{aligned}\Pi_0^{SPA} &= \left[ \int_0^1 t_1^2 dt_1 + \int_0^1 t_2^2 dt_2 \right] - 2 \left[ \frac{2}{3} - \frac{(1+2\phi)(1-\phi)}{2(1+\phi)} \right] \\ &= \frac{(1+2\phi)(1-\phi)}{(1+\phi)} - \frac{2}{3}\end{aligned}$$

and the corresponding average sale price becomes

$$\rho_0^{SPA} = \frac{(1+2\phi)(1-\phi)}{(1-2\phi)(1+\phi)} - \frac{2}{3(1-2\phi)}$$

which completes the proof. ■

**Proof of Proposition 5.1.** Consider the symmetric case. Substituting  $\phi_1 = \phi_2 = \phi$  into (15), and using Lemma 5.1, we can state that

$$\rho_0^{SN} = \frac{32\phi^2 - 56\phi + 25}{64(1-\phi)^3} > \rho_0^{SPA} \geq \rho_0^{FPA}$$

where the second inequality is strict when  $\phi > 0$ , and follows from Proposition 3 in Ettinger (2002).

Consider now the asymmetric case. Lemma 5.1 and the substitution of  $\phi_1 > \phi_2 = 0$  into (15) yields

$$\rho_0^{SN} = \frac{1}{16} \left[ \frac{25}{4(1-\phi_1)} + \phi_1 \right] > \rho_0^{SPA} > \rho_0^{FPA}$$

where the last inequality holds as overbidding is not present in the first-price auction. ■